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Quasi-Heyting Algebras: A New Class of Lattices

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Quasi-Heyting algebras (QHAs) generalize both the Heyting algebras (HAs) of intuitionistic logic and the orthomodular lattices (OMLs) of quantum logic. As in HAs, negation is a Galois connection, which expresses abandonment of the law of the excluded middle, and as in OMLs, incompatibility of propositions is expressed by departures from distributivity. Formulating an equational definition of QHAs leads to generalizations of familiar operations. QHAs are the truth-value objects of a generalization of toposes. So far, this development has aimed to provide foundations of logic and model theory suitable for addressing computer science problems, but they also appear applicable as formulations of the logic of some types of scientific measurement. Many properties of OMLs are likely to have generalizations to QHAs.

1. INTRODUCTION

The development of QHAs began with the observation that the process of pasting Boolean algebras (BAs) so that they would be the blocks of an orthomodular lattice (OML) could be generalized to pasting HAs.

A convenient source of finite examples was available, since finite HAs are the open-set lattices of topologies on finite sets. It was easy to observe that as OML pasting is an amalgamated coproduct in which isomorphic sections are identified; analogous amalgamated coproducts of finite HAs exist, using sections in which all relative pseudocomplements agree, as well as meets, joins, 0, and 1.

This conclusion implies that pasting by isomorphic substructures could be applied to a wide range of types of lattices and algebras, but raises the question whether there would be any value in doing so beyond the satisfaction of curiosity in exploring generalizations of the OML pasting process. For

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the case of pasting HAs it was eventually recognized that if the result Q were a lattice, it would have two properties:

1. The HAs pasted to form Q would be a block structure that would express incompatible operations, as in OMLs.
2. The negation operation would be weakened from an orthocomplementation to a Galois connection, representing a constructive system without the law of the excluded middle, as in intuitionism (i.e., $\neg a \vee a \leq 1$, for $a \in Q$).

Incompatibility of operations is seen both in quantum mechanics and in thermodynamically irreversible systems, such as living organisms and many inorganic systems. In particular, incompatibility arises in several aspects of computation, including both the assignment process in imperative languages, in which previous memory contents are destroyed, and in plausible, but incompatible extensions of a knowledge base in artificial intelligence.

Hence, this new class of lattices opens the possibility of reasoning about computation in a way not imagined before in computer science. They may also be applicable to some aspects of the logic of experimentation in natural science. It has recently been observed that many QHAs are members of the class of difference algebras, but attention has primarily been restricted to lattices to take advantage of the author's previous experience with OMLs (Miller, 1993). The correspondence between QHAs and OMLs is so close that many of the theorems known about OMLs seem likely to extend or generalize to QHAs.

2. AN ALGEBRAIC DEFINITION FOR QHAs

In order not to be restricted to lattices formed by pasting a known collection of HAs together, an algebraic definition was developed by making analogies to some aspects of both HAs and OMLs. Recall first that in an HA H the relative pseudocomplementation operation $\Rightarrow$ is defined for each $a \in H$ by treating H as a category in which the morphisms are defined by the order on H and making the mappings $a \wedge \_\_: H \rightarrow H$ and $a \Rightarrow \_\_:$ $H \rightarrow H$ endofunctors on H that are left and right adjoint to each other, respectively, so that for any $x, b \in H$, $a \wedge x \leq b$ iff $x \leq a \Rightarrow b$. Recall also that in an OML O the compatibility relation C between any $a, b \in O$ is encoded in the Sasaki projection function, $\phi_a(b) := (b \vee a') \wedge a$, so that a C b iff $\phi_a(b) = a \wedge b$. These observations led to the following definition.

Definition 1. A *quasi-Heyting algebra* Q is a bounded lattice with additional mappings $a \Rightarrow$, $\phi_a(\_\_)$: $Q \rightarrow Q$ for each $a \in Q$ such that for $x, b \in Q$:

1. $\phi_a(x) \leq b$ iff $x \leq a \Rightarrow \mathrm{b}$
2. $a \wedge b \leq \phi_a(b) \leq a$

3. $a \wedge b \leq a \Rightarrow b$
4. $a \wedge b = (a \Rightarrow b) \wedge \phi_b(a)$

As in an OML, we use the mapping ϕ_a to define compatibility: $a\ C\ b$ iff $\phi_a(b) = a \wedge b$.

Thus, the definition of a Sasaki projection function in a QHA extends the adjointness between the meet and the relative pseudocomplement mappings in an HA. As in an HA, we define $\neg a := a \Rightarrow 0$. Condition 4 enforces the symmetry of the compatibility relation, which holds in OMLs. Proposition 1 follows immediately from Definition 1.

Proposition 1. In any QHA Q, $a\ C\ a \Rightarrow b$; and $a \Rightarrow b = a \Rightarrow (a \wedge b)$.

However, it is not immediately evident that if a lattice P results from pasting a set of HAs H_i (for $i \in I$ for an index set I), then P satisfies the definition of a QHA. First, the process of pasting on isomorphic sections does not show how to evaluate $a \Rightarrow b$ when there is no H_i such that $a,b \in H_i$, and, second, it does not refer to the Sasaki projections. Evidently, it is consistent with Definition 1 first to extend the $\Rightarrow$ connective to arbitrary $a,b \in P$ by defining $a \Rightarrow b := a \Rightarrow (a \wedge b)$, and then to define $\phi_a(b)$ as $\phi_a(b) := \wedge\{x \in Q \mid b \leq a \Rightarrow x\}$, for P in which this meet always exists. Nevertheless, some additional effort is needed to see that these definitions make a QHA out of a lattice that is pasted collection of HAs supporting the Sasaki projection definition, and the details will appear in Miller (n.d.).

Definition 1 makes an HA of each maximal subalgebra of mutually compatible elements of a QHA, providing the desired analogy with decomposing an OML into Boolean blocks. Then it is immediate that an HA is a QHA in which all pairs of elements are compatible. Likewise, an OML O satisfies Definition 1 when the QHA operations on O are given by $\neg a := a'$, $\phi_a(b) := a \wedge (a' \vee b)$, and $a \Rightarrow b := a' \vee (a \vee b)$ for all $a, b \in$ O. Figure 1 illustrates the resulting containment relations.

Although QHAs have many similarities to OMLs, Zorn's Lemma implies Theorem 1, which has no analogue for the more restricted classes in Fig. 1.

Theorem 1. Every bounded lattice L can be given a QHA structure in which the blocks are the maximal chains of L.

This is because every bounded chain can be given a HA structure (Miller, n.d.). If more than one QHA structure can be imposed on L, Theorem 1 gives the "finest," in the sense of having the smallest blocks.

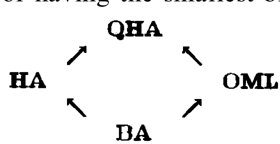

Fig. 1. Containment relations between lattice classes.

3. COMPLETENESS FOR IMPLICATION IN QHA LOGIC

Kalmbach (1983, pp. 238–241) proves that among the five OML polynomials in two variables (denoted $a \to_i b$, $1 \leq i \leq 5$) that satisfy $a \to_i b = 1$ iff $a \leq b$ (the Birkhoff–von Neumann condition for an implication connective), only the OML logic based on $\to_1$ is complete. In this section $\to_1$ is extended to QHAs.

In most cases negation does not behave enough like orthocomplementation for $\neg a$ simply to be substituted for a' in OML formulas to generalize them to QHAs; this is true for $\to_1$. Thus, we generalize

$$a \to_1 b := (a' \wedge b) \vee (a' \wedge b') \vee (a \wedge (a' \vee b))$$

in OMLs by avoiding the use of $\neg$ and defining

$$a \Rightarrow_1 b := ((a \vee b) \wedge (a \vee (b \Rightarrow a))) \Rightarrow \phi_a(b)$$

in QHAs.

The completeness argument follows from observing that, at least in the chain-finite case, the proof in Kalmbach (1983) is greatly simplified by observing that in an OML Q, for any $a, b \in O$, $a \wedge (a \to_1 b) = \phi_a(b)$, but $a \wedge (a \to_i b) = a \wedge b$, for $2 \leq i \leq 5$, so that applying modus ponens with $\to_1$ is the only way to prove all true propositions. Thus, $\Rightarrow_1$ is constructed so that $a \wedge (a \Rightarrow_1 b) = \phi_a(b)$ in QHAs. Again, details will appear in Miller (n.d.).

4. HULLS

There is a further link between QHAs and OMLs. Priestley (1970a, b) proved that every HA H has a Boolean hull: the smallest Boolean algebra into which H can be injected, constructed from an ordered totally disconnected topological space of functions on H. From this one can prove that there is an OML hull O for any QHA Q such that the blocks of O are the hulls of the blocks of Q, and the intersections of the blocks of O are the hulls of the intersections of blocks of Q. Thus, many properties of QHAs follow by inheritance from OML theory.

A simple example appears in Fig. 2, where the bottom square of the cube represents pasting HAs H_1 and H_2 along H_3 to form Q, and likewise B_1 and B_2 are pasted along B_3 to form O, so that the top and bottom squares are pushouts, and all vertical arrows are maps into the respective hulls.

5. TRUTH-VALUE OBJECTS AND LAMINATIONS

Finally, Stout (1979) found a pasting process for toposes generalizing the process for lattices, and it pastes truth-value objects. Stout's examples

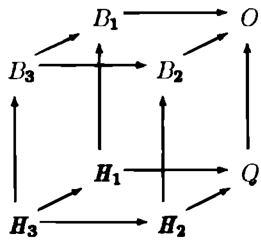

Fig. 2. Boolean and OML hulls of HAs and QHA.

pasted toposes with Boolean truth-value lattices to get objects with OML truth-value lattices, but the general case gives QHA truth-value lattices. Hence there is a possibility of developing new and more widely useful areas of model theory.

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